

Report S 299

On K.L. Chung's problem of imbedding
a time-discrete Markov chain in a
time-continuous one for finitely many states I

by

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Call a stochastic matrix (i.e. a square matrix of order n with elements $p_{jk} \geq 0$ for all $j, k \in \{1, 2, \dots, n\}$ and $\sum_{k=1}^n p_{jk} = 1$ for all $j \in \{1, 2, \dots, n\}$) a P-matrix. Then the following problem (further called Chung's problem) is a specialization of a problem posed in CHUNG [1958] and CHUNG [1960] .

Chung's problem: Find conditions in order that to a given P-matrix P_1 there exists a matrix function $P(t)$ such that $P(t)$ is a P-matrix for each $t \geq 0$, for which ¹⁾

$$(1) \quad \left\{ \begin{array}{l} \lim_{t \downarrow 0} P(t) = I, \\ P(s+t) = P(s)P(t) \quad \text{for all real } s, t > 0, \\ P(1) = P_1. \end{array} \right.$$

For $n=2$ Chung's problem has been solved by FRÉCHET [1952] and BELLMAN [1960]. In my thesis (RUNNENBURG [1960]) the case $n=3$ was considered.

It is well-known (cf. DOOB [1953] , BIRKHOFF and VARGA [1958]), that if $P(t)$ exists, then

$$(2) \quad P(t) = e^{Q_1 t} \quad \text{for all real } t \geq 0,$$

where Q_1 is a Q-matrix (i.e. a square matrix of order n with elements q_{jk} satisfying $q_{jk} \geq 0$ if $j \neq k$ for all $j, k \in \{1, 2, \dots, n\}$ and $\sum_{k=1}^n q_{jk} = 0$ for all $j \in \{1, 2, \dots, n\}$).

Furthermore, if $\mu_1, \mu_2, \dots, \mu_n$ are the eigenvalues of Q_1 , then the eigenvalues $\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t)$ of $P(t)$ are given by

$$(3) \quad \lambda_j(t) = e^{\mu_j t} \quad \text{for all real } t \geq 0 \text{ and all } j.$$

1) I is the $n \times n$ identity matrix.

According to GANTMACHER [1959] , in 1938 Kolmogorov posed the problem: Characterize the complex numbers z with $|z| \leq 1$ which can occur as eigenvalues of an n^{th} order stochastic matrix. This problem was partly solved in DMITRIEV and DYNKIN [1946] and definitely in KARPELEWITSCH [1951] . Making use of their results, the next theorem can be proved.

Theorem: The eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ of a Q-matrix (of order $n \geq 3$) satisfy

$$(4) \quad \left(\frac{1}{2} + \frac{1}{n}\right)\pi \leq \arg \mu_j \leq \left(\frac{3}{2} - \frac{1}{n}\right)\pi \quad \text{for } 1 \leq j \leq n;$$

the only Q-matrices Q_1^* (with elements q_{jk}^*) with at least one $\mu_j \neq 0$ on the boundary of this region are given (after a suitable renumbering of states, i.e. rows and columns at the same time) by

$$(5) \quad q_{jk}^* = \begin{cases} -\alpha & \text{for } j \equiv k \pmod{n}, \\ \alpha & \text{for } j \equiv k-1 \pmod{n}, \\ 0 & \text{otherwise,} \end{cases}$$

where α is an arbitrary positive number.

Remark 1: From this theorem we conclude, that for $n \geq 3$ all eigenvalues $\lambda_j(t)$ of $P(t) = e^{Q_1^* t}$ satisfy

$$(6) \quad \lambda_j(t) \in H_n,$$

where H_n is a heart-shaped region in the complex plane, contained in the unit circle and symmetric with respect to the real axis, with the curve

$$(7) \quad \exp\left(-1 + \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}\right)t \quad \left(\text{where } 0 \leq t \leq \frac{\pi}{\sin \frac{2\pi}{n}}\right)$$

as boundary in the region $\text{Im } z \geq 0$. Hence Chung's problem can only be solved for those P_1 for which all eigenvalues belong to the region H_n . The only matrices $e^{Q_1^*}$ with a Q-matrix Q_1 for which at least one eigenvalue $\lambda \neq 1$ lies on the boundary curve of the region H_n are given by

$$(8) \quad P_1^* = e^{Q_1^*}$$

Proof of the theorem: If P_1 is a P-matrix, then $Q_1 = P_1 - I$ is a Q-matrix; if Q_1 is a Q-matrix, then $P_{1,\beta} = I + \beta Q_1$ (where $\beta > 0$ is chosen in such a way that $0 \leq 1 + \beta q_{jj} \leq 1$ for $1 \leq j \leq n$, e.g. $\beta = -\left(\min_{1 \leq j \leq n} q_{jj}\right)^{-1}$ if some $q_{jj} < 0$) is a P-matrix. Hence if for an arbitrary Q-matrix Q_1 we have

$$(9) \quad \det(Q_1 - \mu I) = 0,$$

then

$$(10) \quad \det(P_{1,\beta} - (\beta\mu + 1)I) = 0.$$

Now Karpelewitsch has shown that the roots of the characteristic equation

$$(11) \quad \det(P_1 - \lambda I) = 0$$

for an arbitrary P-matrix P_1 , satisfy

$$(12) \quad \left(\frac{1}{2} + \frac{1}{n}\right)\pi \leq \arg(\lambda - 1) \leq \left(\frac{3}{2} - \frac{1}{n}\right)\pi$$

for $n \geq 3$, and so with $\lambda = \beta\mu + 1$, because of $\beta > 0$ we have

$$(13) \quad \left(\frac{1}{2} + \frac{1}{n}\right)\pi \leq \arg \mu \leq \left(\frac{3}{2} - \frac{1}{n}\right)\pi.$$

If Q_1^* has an eigenvalue μ_j for which $\arg \mu_j = \left(\frac{1}{2} + \frac{1}{n}\right)\pi$ with $\mu_j \neq 0$, i.e. $\mu_j = \gamma(e^{\frac{2\pi i}{n}} - 1)$ with a $\gamma > 0$. then $P_{1,\beta}^*$ has an eigenvalue λ_j with $\arg(\lambda_j - 1) = \left(\frac{1}{2} + \frac{1}{n}\right)\pi$ and $\lambda_j - 1 \neq 0$. As Dmitriev and Dynkin have shown, in that case the elements p_{jk}^* of $P_{1,\beta}^*$ satisfy (after a suitable renumbering of the rows and columns at the same time)

$$(14) \quad p_{jk}^* = \begin{cases} 1 - \alpha_j^* & \text{for } j \equiv k \pmod{n}, \\ \alpha_j^* & \text{for } j \equiv k-1 \pmod{n}, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \leq \alpha_j \leq 1$ for each $j \in \{1, 2, \dots, n\}$. Therefore the elements q_{jk}^* of Q_1^* are given by

$$(15) \quad q_{jk}^* = \begin{cases} -\alpha_j & \text{for } j \equiv k \pmod{n} \\ \alpha_j & \text{for } j \equiv k-1 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

where we have written α_j instead of $\frac{\alpha_j}{\alpha_j}$, and so $\alpha_j \geq 0$ for each $j \in \{1, 2, \dots, n\}$.

For the characteristic equation of Q_1^* we obtain

$$(16) \quad (\alpha_1 + \mu)(\alpha_2 + \mu) \dots (\alpha_n + \mu) = \alpha_1 \alpha_2 \dots \alpha_n.$$

In order that $\mu = \gamma(e^{\frac{2\pi i}{n}} - 1)$, with a constant $\gamma > 0$, is a root of this equation, all α_j must be positive. We shall prove that

$$(17) \quad \alpha_j = \gamma \quad \text{for all } j \in \{1, 2, \dots, n\},$$

for which it is sufficient to consider $\gamma = 1$.

If we introduce the finite positive numbers $\theta_j = \frac{1}{\alpha_j}$ in (16) and substitute $e^{\frac{2\pi i}{n}} - 1$ for μ , we obtain

$$(18) \quad \left\{ \theta_1 e^{\frac{2\pi i}{n}} + (1 - \theta_1) \right\} \left\{ \theta_2 e^{\frac{2\pi i}{n}} + (1 - \theta_2) \right\} \dots \left\{ \theta_n e^{\frac{2\pi i}{n}} + (1 - \theta_n) \right\} = 1.$$

Consider a line in the complex plane which does not pass through the origin. Let $r(\varphi)$ be the absolute value of the complex number on this line with argument φ . Then $\log r(\varphi)$ is a strictly convex analytic function of φ for finite $r(\varphi)$. Hence if $\varphi_1 \neq \varphi_2$, we have for any $p_1, p_2 > 0$ with $p_1 + p_2 = 1$

$$(19) \quad p_1 \log r(\varphi_1) + p_2 \log r(\varphi_2) > \log r(p_1 \varphi_1 + p_2 \varphi_2).$$

Therefore

$$(20) \quad r(\varphi_1) r(\varphi_2) \dots r(\varphi_n) > r\left(\frac{\varphi_1 + \varphi_2 + \dots + \varphi_n}{n}\right)^n,$$

unless $\varphi_1 = \varphi_2 = \dots = \varphi_n$.

Now consider for $n \geq 3$ the points $\theta_j \cdot e^{\frac{2\pi i}{n}} + (1-\theta_j) \cdot 1$ for finite $\theta_j > 0$, which lie on the line passing through 1 and $e^{\frac{2\pi i}{n}}$. Then $0 < \varphi_j < (\frac{1}{2} + \frac{1}{n})\pi$. Under the restriction $\varphi_1 + \varphi_2 + \dots + \varphi_n \equiv 0 \pmod{2\pi}$, the product $r(\varphi_1)r(\varphi_2)\dots r(\varphi_n)$ assumes its smallest value for $\varphi_1 = \varphi_2 = \dots = \varphi_n = \frac{2\pi}{n}$. Any other choice of values for the φ_j leads to a larger value for the product. Hence if we introduce polar coordinates in (18) by writing

$$(21) \quad \theta_j e^{\frac{2\pi i}{n}} + 1 - \theta_j = r(\varphi_j) e^{i\varphi_j},$$

we find that under the condition "all θ_j are finite positive numbers" the left-hand side of (18) has smallest positive value 1, where the value 1 is only obtained if $\varphi_j = \frac{2\pi}{n}$ for all j . Therefore, if equation (18) holds, we must have $\theta_j = 1$ for all j or $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$.

Remark 2: For $n=3$ one may easily verify that there exist P -matrices P_1 for which Chung's problem does not have a solution, although P_1 has all eigenvalues inside H_3 . If P_1 is a circulant (i.e. a square matrix of order n with elements $p_{jk} = p_{k-j}$, where $k-j$ is taken modulo n), then for $n=3$ a circulant Q_1 can be found which is a Q -matrix and for which $P_1 = e^{Q_1}$ holds, on condition that all eigenvalues of P_1 are inside H_3 .

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Report S 299

On K.L. Chung's problem of imbedding
a time-discrete Markov chain in a
time-continuous one for finitely many states II^{*)}

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*) Part I of this report was written before we knew of KINGMAN [5]. The problem is really due to G. ELFVING [6].

1

The purpose of this note is to show that one of the results obtained in the previous one [1], can be deduced without using the work of DMITRIEV and DYNKIN [2] and KARPELEWITSCH [3]. For a statement of the problem we refer to [1].

We use the fact that a Markov-chain on a finite set consisting of n states $A_1, A_2, \dots, A_n$, induces an abelian semi-group G of real linear transformations on an R_n : with the Markov matrix (p_{jk}) (where $p_{jk} = P\{x_{n+1} = A_k | x_n = A_j\}$) corresponds the linear transformation π defined by

$$\pi e_j = (p_{j1}, p_{j2}, \dots, p_{jn}),$$

where e_j is the j 'th unit vector.

A necessary and sufficient condition that the matrix of a given linear transformation φ is a Markov matrix is

$$(1) \quad \varphi W \subset W,$$

where W is the set of all probability distributions on $\{A_1, A_2, \dots, A_n\}$

$$(2) \quad W = \{x | x = (\xi_1, \xi_2, \dots, \xi_n); \forall_j \xi_j \geq 0; \sum_{j=1}^n \xi_j = 1\}.$$

2

Now let G be a continuous one-parameter abelian semi-group of linear transformations satisfying (1) (ϵ is the identical transformation on R_n)

$$G = \{\varphi_t | t \in (0, \infty); \forall_t^{(0, \infty)} \varphi_t W \subset W; \forall_t^{(0, \infty)} \forall_s^{(0, \infty)} \varphi_t \varphi_s = \varphi_{t+s}; \lim_{t \downarrow 0} \varphi_t = \epsilon\}.$$

It is well-known (cf e.g. FRÉCHET [4]) that under these conditions the eigenvalues of φ_t are $1, \lambda_1^t, \lambda_2^t, \dots, \lambda_m^t$ ($m \leq n-1$), where $1, \lambda_1, \lambda_2, \dots, \lambda_m$ are eigenvalues of φ_1 , for some choice of the arguments of $\lambda_1, \dots, \lambda_m$. Moreover there is at least one vector $p \in W$, which is invariant under G .

Let $\lambda = \xi e^{i\psi}$ (ξ real and ≥ 0) be an eigenvalue of φ_1 . We suppose that ψ is chosen in such a way that $\xi^t e^{it\psi}$ is, for all $t > 0$, an eigenvalue of φ_t . Moreover we suppose for the moment $\psi \neq 0$; $\xi \neq 0$ and $n \geq 3$. (The inequality (20) which

we want to establish is trivially satisfied for $\psi=0$ or $\varphi=0$).

Let

$$z_0 = x_0 + iy_0,$$

where x_0 and y_0 are real vectors, be an eigenvector of φ_1 belonging to the eigenvalue λ . As $\psi \neq 0$, it follows that x_0 and y_0 are independent. Then the two-dimensional subspace V spanned by x_0 and y_0

$$V = \{ u | u = \alpha x_0 + \beta y_0; \alpha, \beta \text{ real} \}$$

is invariant under G , and the movement induced by G in V is easily seen to be described by the equations

$$(3) \quad \begin{cases} \varphi_t x_0 = \varphi^t (x_0 \cos t\psi - y_0 \sin t\psi), \\ \varphi_t y_0 = \varphi^t (x_0 \sin t\psi + y_0 \cos t\psi). \end{cases}$$

Introducing

$$W_0 = \{ x | x \in W; \exists_j \xi_j = 0 \},$$

$$U = W \cap (p+V),$$

$$U_0 = W_0 \cap (p+V),$$

the condition (1) together with the invariance of p and V yields

$$(4) \quad V_t^{(0, \infty)} \varphi_t U \subset U.$$

As U is a closed convex polygon¹⁾ in the plane $p+V$ and

1) It might seem that this statement is false for $n=3$. However in that case our hypothesis $\psi \neq 0$ entails that $1, \varphi e^{it\psi}$ and $\varphi e^{-it\psi}$ are the only eigenvalues of φ_t (and hence of the adjoint transformation). Therefore $\{ x | \xi_1 + \xi_2 + \xi_3 = 0 \}$ is the only possible real invariant two-dimensional subspace and thus $V = \{ x | \xi_1 + \xi_2 + \xi_3 = 0 \}$, implying $W \subset p+V$, whence $U=W$ and $U_0=W_0$.

U_0 its boundary the condition (4) is satisfied if and only if

$$(5) \quad \forall_t^{(0, \infty)} \varphi_t U_0 \subset U$$

is satisfied.

To derive a geometric condition equivalent with (5), consider a vector $w \in U_0$. The curve $\{\varphi_t w | t > 0\}$ must, on account of (5), be in U . Hence a vector v_1 tangent to $\{\varphi_t w | t \geq 0\}$ in w must point from w to the interior of U . To make this condition more precise consider a vector v_2 pointing from w along U_0 in such a way that v_1 lies inside the angle¹⁾ formed by $p-w$ and v_2 .

If now we define

$$(6) \quad \theta_w = \angle(p-w, v_1); \quad \theta_{o,w} = \angle(p-w, v_2),$$

then it is easily seen that (5) is satisfied if and only if

$$(7) \quad \forall w \in U_0 \quad \theta_w \leq \theta_{o,w}$$

is true.

3 It does not seem possible to deduce useful results directly from (7), as θ_w is in general a complicated expression depending on w . The only case in which θ_w is independent of w obtains when $|x_0| = |y_0|$ and the inner product $(x_0, y_0) = 0$, which will be shown below. The general case can be reduced to this case by means of the following considerations:

Let σ be a real non-singular linear transformation on R_n . Then the semi-group

$$G^* = \sigma G \sigma^{-1}$$

1) There is a slight difficulty here on account of the fact that p may be in U_0 . This case does not occur if $\psi \neq 0$, $\varepsilon \neq 0$: On account of (3) any vector $w-p$ performs a complete rotation around p in the time interval $(0, 2\pi/\psi]$. Hence (5) and $p \in U_0$ are inconsistent.

consists of those linear transformations φ_t^* for which

$$\varphi_t^* = \sigma \varphi_t \sigma^{-1}.$$

These φ_t^* have the same eigenvalues as φ_t and eigenvectors

$$z^* = \sigma z, \text{ with } z^* = x^* + iy^*; \quad x^* = \sigma x; \quad y^* = \sigma y,$$

where $z = x + iy$ is an eigenvector of φ_t .

Application of the transformation σ to $W, p, x_0, y_0, V, U, U_0, v_1, v_2$ yields $W^*, p^*, x_0^*, y_0^*, V^*, U^*, U_0^*, v_1^*, v_2^*$, i.e. $W^* = \sigma W$, etc.

We can now repeat the argument of §2 and arrive at equivalent conclusions. In particular, if we define

$$(6^*) \quad \theta_w^* = \angle(p^* - w, v_1^*); \quad \theta_{0,w}^* = \angle(p^* - w, v_2^*),$$

then

$$(7^*) \quad \forall w \in U_0^* \quad \theta_w^* = \theta_{0,w}^*$$

is equivalent with (7).

In view of the remark at the beginning of this section, we choose σ in such a way that

$$(8) \quad |x_0^*| = |y_0^*| \quad \text{and} \quad (x_0^*, y_0^*) = 0.$$

Using (8) it is easy to calculate θ_w^* . For an arbitrary $w \in p^* + V^*$ we have

$$(9) \quad w = p^* + \alpha(x_0^* \sin \gamma + y_0^* \cos \gamma) \quad (\alpha, \gamma \text{ real}).$$

Consider $\{\varphi_t w \mid t \geq 0\}$. A vector tangent to this curve in w is given by

$$(10) \quad v_1^* = \lim_{h \downarrow 0} \frac{\varphi_h^* w - w}{h}.$$

Substitution of (9) and

$$(3^*) \quad \begin{cases} \varphi_t^* x_0^* = e^{t\psi} (x_0^* \cos t\psi - y_0^* \sin t\psi) \\ \varphi_t^* y_0^* = e^{t\psi} (x_0^* \sin t\psi + y_0^* \cos t\psi) \end{cases}$$

in (10) gives after some calculation

$$(11) \quad v_1^* = -\alpha A \left[x_0^* \sin(\gamma - \chi) + y_0^* \cos(\gamma - \chi) \right],$$

where

$$(12) \quad A^2 = \psi^2 + \log^2 \xi; \quad 0 < \chi < \pi \quad \text{and} \quad \text{ctn } \chi = -\psi^{-1} \log \xi.$$

Then, using (8), (11) and (9), it follows that

$$(13) \quad \cos \theta_w^* = \frac{(p^* - w, v_1^*)}{|p^* - w| \cdot |v_1^*|} = \cos \chi,$$

and hence (as θ_w^* and χ are both in $[0, \pi]$)

$$(14) \quad \theta_w^* = \chi.$$

We thus see that θ_w^* is indeed independent of w . Our condition (7*) yields

$$(15) \quad \chi \leq \inf_{w \in U_0^*} \theta_{0,w}^*.$$

4

As σ is non-singular, W^* is like W an $(n-1)$ -simplex. Hence U_0^* , being the boundary of the intersection of W^* with a plane that has at least one point (namely p^*) in common with W^* , is a convex polygon with at most n vertices, say $a_1, a_2, \dots, a_n$. It is then easy to see, that the infimum in the right-hand side of (15) is reached if w is in one of the vertices of U_0^* . Therefore, if we introduce ¹⁾

$$\left. \begin{aligned} \alpha_j &= \angle p^* a_j a_{j+1} \\ \alpha'_j &= \angle p^* a_j a_{j-1} \end{aligned} \right\} \quad (j=1, 2, \dots, n; \text{ indices taken mod } n),$$

then

$$(16) \quad \inf_{w \in U_0^*} \theta_{0,w}^* = \min_j \alpha_j \quad \text{or} \quad \min_j \alpha'_j,$$

1) For any three vectors a, b and c we use the following notation:
 $\angle abc \stackrel{\text{df}}{=} \angle (a-b, c-b).$

depending on whether the sense of rotation in p^*+V^* given by p^*, w, v_1^* in that order is the same as the rotational sense given by $a_1, a_2, \dots, a_n$ or that given by $a_n, a_{n-1}, \dots, a_1$ respectively.

In 5 we shall give a proof of the fact that

$$(17) \quad \min_j \alpha_j \leq \frac{n-2}{2n} \pi ; \min_j \alpha'_j \leq \frac{n-2}{2n} \pi ,$$

with equality in at least one of these two relations if and only if the polygon under consideration is regular and p^* its centre.

Combination of (15), (16) and (17) yields

$$\chi \leq \frac{n-2}{2n} \pi ,$$

from which it follows that

$$(18) \quad \xi \leq e^{-\psi} \operatorname{tg} \pi/n .$$

Now ψ is of the form $\psi'_0 + 2k\pi$ with $0 \leq \psi'_0 < 2\pi$ and k an integer. As $\bar{\lambda}$ is also an eigenvalue of φ_1 , we may conclude

$$(19) \quad \exists_k \quad \xi \leq e^{-(\psi'_0 + 2k\pi) \operatorname{tg} \pi/n} \quad \text{and} \quad \xi = e^{-(\psi'_0 - 2k\pi) \operatorname{tg} \pi/n} .$$

If we put

$$\psi_0 = \min \{ \psi'_0, 2\pi - \psi'_0 \} ,$$

than either $\psi'_0 + 2k\pi \geq \psi_0$ or $-\psi'_0 - 2k\pi \geq \psi_0$, hence

$$(20) \quad \xi \leq e^{-\psi_0 \operatorname{tg} \pi/n} .$$

If $\psi=0$, then $\psi_0=0$ also and as $\xi \leq 1$ always, (20) is trivially satisfied in this case. The restriction $\xi \neq 0$ can now be removed also, as again (20) is satisfied in this case. Our final result is therefore: if λ is an eigenvalue of φ_1 , ξ its absolute value, ψ_0 the minimum value of the non-negative arguments of λ and $\bar{\lambda}$, then (20) must be satisfied. This result is equivalent to (6) in [1].

5 It remains to prove (17). To this purpose suppose, with the notation of section 4

$$(21) \quad \forall_j \alpha_j \geq \frac{n-2}{2n} \pi.$$

Then it is possible to find vectors¹⁾ $p_1 \in \langle p^*, a_1 \rangle, p_2 \in \langle p^*, a_2 \rangle, \dots, p_n \in \langle p^*, a_n \rangle$ in such a way that
 $\angle p^* a_n p_1 = \angle p^* p_1 p_2 = \dots = \angle p^* p_{n-1} p_n = \frac{n-2}{2n} \pi.$

Then, as $p_n \in \langle p^*, a_n \rangle$

$$\frac{|p_n - p^*|}{|a_n - p^*|} \leq 1,$$

with equality if and only if $\forall_j p_j = a_j$. Introducing $\alpha_j'' = \angle p^* p_j p_{j-1}$ ($j=2, \dots, n$) and $\alpha_1'' = \angle p^* p_1 a_n$, we find

$$\begin{aligned} 1 &\geq \frac{|p_n - p^*|}{|a_n - p^*|} = \frac{|p_n - p^*|}{|p_{n-1} - p^*|} \cdot \frac{|p_{n-1} - p^*|}{|p_{n-2} - p^*|} \cdot \dots \cdot \frac{|p_1 - p^*|}{|a_n - p^*|} = \\ &= \prod_{j=1}^n \frac{\sin \frac{n-2}{2n} \pi}{\sin \alpha_j''}. \end{aligned}$$

Hence

$$(22) \quad \prod_{j=1}^n \sin \alpha_j'' \geq \sin^n \frac{n-2}{2n} \pi,$$

with equality if and only if $\forall_j \alpha_j'' = \alpha_j'$.

On the other hand, as the sum of the angles ($n\pi$) of the n triangles $\langle a_n, p_1, p^* \rangle, \langle p_1, p_2, p^* \rangle, \dots, \langle p_{n-1}, p_n, p^* \rangle$ is equal to $\sum_{j=1}^n \alpha_j'' + 2\pi + n \cdot \frac{n-2}{2n} \pi$, we have

$$\sum_{j=1}^n \alpha_j'' = n \cdot \pi - 2\pi - \sum_{j=1}^n \frac{n-2}{2n} \pi = \frac{n-2}{2} \pi.$$

As $-\log \sin x$ is a convex function for $0 < x < \pi$, we have

 1) We denote by $\langle b_1, b_2, \dots, b_k \rangle$ the closed convex hull of $\{b_1, \dots, b_k\}$: $\langle b_1, b_2, \dots, b_k \rangle = \{x | x = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_k b_k;$

$$\forall_j \alpha_j \geq 0; \sum_{j=1}^k \alpha_j = 1\}.$$

$$(23) \quad \prod_{j=1}^n \sin \alpha_j'' \leq \sin^n \frac{1}{n} \sum_{j=1}^n \alpha_j'' = \sin^n \frac{n-2}{2n} \pi ,$$

with equality if and only if $\forall_j \alpha_j'' = \frac{n-2}{2n} \pi$.

Combination of (22) and (23) yields:

$$(24) \quad \forall_j \alpha_j' = \alpha_j'' = \frac{n-2}{2n} \pi .$$

We can now repeat this argument, starting from

$$(25) \quad \forall_j \alpha_j' \geq \frac{n-2}{2n} \pi ,$$

which follows from (24). We then arrive at the conclusion

$$(26) \quad \forall_j \alpha_j = \frac{n-2}{2} \pi .$$

Hence, either of (21) or (25) entails (24) and (26).
In other words: Either

$$\forall_j \alpha_j = \alpha_j' = \frac{n-2}{2n} \pi ,$$

in which case $\langle a_1, a_2, \dots, a_n \rangle$ is regular with centre p^* ,
or

$$\min_j \alpha_j < \frac{n-2}{2n} \pi \quad \text{and} \quad \min_j \alpha_j' < \frac{n-2}{2n} \pi ,$$

which proves (17).

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